ON HYDRODYNAMIC PHENOMENA ACCOMPANYING MELTING IN A PARTICULAR CASE

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The problem of melting ice filling a lower half-space, under the effect of a heavy heated cylinder of sufficiently large radius is considered. An analytic solution is obtained for a linear formulation of the nonstationary problem of motion of the fluid being formed because of melting of the solid phase.

A hot normal cylinder is placed on ice. Under the cylinder the ice thaws and a liquid layer is formed between the cylinder whose temperature is kept constant equal to T_0 and the ice, and its spreads to the side under the effect of the cylinder weight. It is assumed that the thickness of the liquid layer h is much less than the cylinder radius r_0, which permits neglecting edge effects.

Let us introduce a cylindrical r, φ, z coordinate system by directing the z axis upward opposite to the acceleration of gravity $\mathbf{g} = g\boldsymbol{\gamma}$ and superposing the x = 0 plane on the horizontal ice surface and the lower base of the heated cylinder at the initial instant. With respect to the laboratory reference system connected to the fixed mass of ice, the selected coordinate system moves downward at a velocity v_0 equal to the steady velocity of the cylinder so that the ice surface corresponds to the coordinate $z_0(t)$, and the lower cylinder surface to $-z_0(t) + h(t)$, where z_0 and the thickness of the fluid layer h are functions of the time t.

The system of equations, boundary, initial, and integral conditions corresponding to the formulated problem is:

$$\dot{\mathbf{v}} + (\mathbf{v}\nabla)\mathbf{v} = -\frac{\nabla p}{\rho} + \nu\Delta\mathbf{v} - \mathbf{g}\beta T, \quad \dot{T} + \mathbf{v}\nabla T = \chi\Delta T; \qquad (1)$$

$$\nabla\mathbf{v} = 0;$$

$$z = z_0(t), \quad v_z = v_0, \quad v_r = 0, \quad T = 0; \qquad (2)$$

$$z = z_0(t) + h(t), \quad v_z = \dot{z}_0 + \dot{h}, \quad v_r = 0;$$

$$T = T_0, \quad \varkappa\frac{\partial T}{\partial z} = \rho L(v_0 - \dot{z}_0);$$

$$r = r_0, \quad p = 0; \quad t = 0, \quad z_0 = 0, \quad h = 0; \qquad (3)$$

$$\int_{z_0}^{z_0+h} v_r dz = \frac{r}{2}\, v_0 - (\dot{z} + \dot{h}); \qquad (4)$$

$$mg - \int_{S_0} pdS = m(\ddot{z} + \ddot{h}). \qquad (5)$$

The boundary conditions (2) on the moving ice surface $z_0(t)$ take account of the adhesion condition ($v_r = 0$), the constancy of the solid phase temperature at the melting point (T = 0), and the velocity of the coordinate system ($v_z = v_0$) with respect to the ice at rest in the laboratory reference system.

The adhesion condition, ($v_r = 0$), the sustained cylinder temperature ($T = T_0$), and the impermeability of the boundary for the fluid (the fluid velocity v_z on the boundary equals the boundary velocity $\dot{z}_0 + \dot{h}$) are given analogously on the lower cylinder surface $z_0(t) + h(t)$.

Zero external pressure is given on the cylinder edge.

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The initial conditions (3) are homogeneous, there is no gap up to the time the hot body makes contact with the ice, and the origin is at the ice surface.

Equation (4) is the integral condition of conservation of fluid momentum in a cylindrical volume of arbitrary radius r.

The last integral condition (5) is the law of cylinder motion under the effect of gravity and a pressure force applied to its lower surface $S_0 = \pi r_0^2$.

Let us examine the axially symmetric solution of the boundary value problem (1)-(5). It then follows from (4) and the continuity equation that

$$v_z = g(z, t), \quad v_r = -\frac{r}{2} g'(z, t), \tag{6}$$

where the prime over the letter will denote, here and henceforth, the derivative with respect to the coordinate z. In this case, the integral equation (4) is satisfied identically by virtue of the boundary conditions (2) and, hence will no longer be written down. The same holds for the continuity equation.

Let us go over the dimensionless quantities by selecting ν/v_0 as the unit of length, v_0 for velocity, ν/v_0 for time, T_0 for temperature, and $\rho v_0^2/2$ for pressure. Denoting the variables by the previous letters and keeping (6) in mind, we obtain

$$\begin{gathered}
\dot{\mathbf{v}} + (\mathbf{v}\nabla)\mathbf{v} = -\frac{1}{2}\nabla p + \Delta\mathbf{v} - GT\gamma, \ P(\dot{T} + \mathbf{v}\nabla T) = \Delta T; \\
z = z_0(t), \quad g = 1, \quad g' = 0, \quad T = 0; \\
z = z_0(t) + h(t), \quad g = \dot{z}_0 + \dot{h}, \quad g' = 0, \quad T = 1, \quad T' = Q(1 - \dot{z}_0); \\
r = R, \quad p = 0; \quad t = 0, \quad z_0 = 0, \quad h = 0; \\
M + \int_{S_0} p dS = N(\ddot{z}_0 + \ddot{h}).
\end{gathered} \tag{7}$$

Here we have introduced the dimensionless parameters

$$G = \frac{g\beta T_0 \nu}{v_0^3}, \quad Q = \frac{\rho L \nu}{T_0 \varkappa}, \quad M = \frac{2mg}{\nu^2\rho}, \quad N = \frac{2mv_0^3}{\nu^3\rho}, \quad P = \frac{\nu}{\chi}$$

and the dimensionless cylinder radius $R = r_0 v_0/\nu$.

Under the assumption of slowness of the motion ($(\mathbf{v}\nabla)\mathbf{v} \ll \Delta\mathbf{v}$) and smallness of the numbers G and P, and after eliminating the pressure from the Navier–Stokes equations, the boundary value problem (7) is

$$\begin{gathered}
\dot{g}^{II} = g^{IV}; \\
z = z_0, \quad g = 1, \quad g' = 0; \\
z = z_0 + h, \quad g = \dot{z}_0 + \dot{h}, \quad g' = 0; \\
t = 0, \quad z_0 = 0, \quad h = 0; \\
M - \int_{S_0} p dS = N(\ddot{z}_0 + \ddot{h}).
\end{gathered} \tag{8}$$

The temperature is linear:

$$T = \frac{z - z_0(t)}{h(t)}, \quad [T' = Q(1 - \dot{z}_0)]. \tag{9}$$

Let us seek the solution of the boundary value problem (8) in the form of the series

$$g = \sum_{n=0}^{\infty} q_n(z)\xi^n, \quad z_0 = \sum_{n=0}^{\infty} A_n\xi^n, \quad h = \sum_{n=0}^{\infty} B_n\xi^n, \tag{10}$$

where $\xi = \exp(-\lambda t)$. The equations

$$\begin{gathered}
q_0 = a_0 + b_0 z + c_0 z^2 + d_0 z^3, \\
q_n = a_n + b_n z + c_n \cos k_n z + d_n \sin k_n z, \quad n \neq 0, \quad k_n = \sqrt{n\lambda},
\end{gathered} \tag{11}$$

are obtained for $q_n(z)$. To determine the pressure p in the boundary conditions of problem (8), let us integrate the projection of the Navier—Stokes equation on the axis r with respect to r and let us satisfy the condition p = 0 on r = R

$$p = \frac{R^2 - r^2}{2}(\dot{g}' - g'''), \tag{12}$$

where g(z, t) is taken from (10).

Let us substitute (10)-(12) into the boundary and integral conditions of the system of equations (8) and let us collect terms with ξ^0. After simple manipulations, equations are obtained to determine a_0, b_0, c_0 and d_0, as well as the steady thickness of the liquid layer and the rate of cylinder drop:

$$v_0 = \left(\frac{2}{3\pi}\cdot\frac{mg}{\rho^4 r_0^4}\cdot\frac{T_0^3\varkappa^3}{L^3\nu}\right)^{1/4}; \quad h = r_0\left(\frac{3\pi}{2}\cdot\frac{\varkappa\nu}{L}\cdot\frac{T_0}{mg}\right)^{1/4}; \tag{13}$$

$$a_0 = 1 - \frac{3A_0^2}{B_0^2} - \frac{2A_0^3}{B_0^3}; \quad b_0 = \frac{6A_0}{B_0^2} + \frac{6A_0}{B_0^3}; \tag{14}$$

$$c_0 = -\frac{3}{B_0^2} - \frac{6A_0}{B_0^3}; \quad d_0 = \frac{2}{B_0^3}; \quad B_0 = \frac{T_0\varkappa}{L\rho\nu}.$$

The B_0 in (14) equals the steady dimensionless layer thickness written in dimensional form in (13). The distance A_0 which the ice—water interface traverses up to the build-up of the stationary mode is found in the next approximation in ξ. Let us derive the equations for this approximation

$$b_0A_1 + c_0 2A_0A_1 + d_0 3A_0^2A_1 + a_1 + b_1A_0 + c_1\cos k_1A_0 + d_1\sin k_1A_0 = 0;$$
$$2c_0A_1 + 6d_0A_0A_1 + b_1 - k_1c_1\sin k_1A_0 + k_1d_1\cos k_1A_0 = 0;$$
$$6(A_1 + B_1) + 2c_0(A_0 + B_0)(A_1 + B_1) + 3d_0(A_0 + B_0)^2(A_1 + B_1) + a_1$$
$$+ b_1(A_0 + B_0) + c_1\cos k_1(A_0 + B_0) + d_1\sin k_1(A_0 + B_0) = -k_1^2(A_1 + B_1);$$

$$2c_0(A_1 + B_1) + 6d_0(A_0 + B_0)(A_1 + B_1) + b_1 - k_1c_1\sin k_1(A_0 + B_0)$$
$$+ k_1d_1\cos k_1(A_0 + B_0) = 0, \quad N(A_1 + B_1)k_1^2 = \frac{\pi}{4}R^4b_1; \tag{15}$$
$$B_1 + k_1^2A_1B_0 = 0.$$

Assuming that we limit ourselves to a linear approximation in ξ, let us append initial conditions, truncated at the second term

$$A_0 + A_1 = 0, \quad B_0 + B_1 = 0, \tag{16}$$

to this system of equations.

Hence, A_1 and B_1 are at once expressed successfully in terms of A_0 and B_0, which in turn permits estimation of A_0 by using the last equation in the system (15)

$$A_1 = -A_0, \quad B_1 = -B_0, \quad A_0 = -k_1^{-2}.$$

The restriction to a term linear in $\xi = \exp(-\lambda t)$ in the expansions (10) means, physically, the assumption of sufficiently large decrements $\lambda \equiv k_1^2$. It is easy to see that B_0 should also be on the order of k_1^{-2} so that we can set $B_0 = \alpha/k_1^2$, where α is a new unknown on the order of one.

To determine the remaining five unknowns a_1, b_1, c_1, d_1 and α there are the five first equations in the system (15). Because of the complexity of the system it is expedient to use the smallness of A_0 and B_0 and to set

$$\sin k_1A_0 \approx \frac{1}{k_1}, \quad \cos k_1A_0 \approx 1, \quad \sin k_1(A_0 + B_0) \approx \frac{1+\alpha}{k_1},$$
$$\cos k_1(A_0 + B_0) \approx 1.$$

In this approximation α is found easily and turns out to equal 3 approximately. This permits finding the "build-up time" of the stationary mode $1/\lambda = k_1^{-2}$ as well as the distance traversed by the cylinder during this time:

$$-A_0 \cong \frac{1}{3}\cdot\frac{T_0\varkappa}{\rho\nu L} \tag{17}$$

(according to the coefficient B_0 already found (see (14)).

NOTATION

T	is the temperature;
v	is the velocity;
p	is the pressure;
t	is the time;
ρ	is the density;
ν	is the coefficient of kinematic viscosity;
χ	is the temperature conductivity;
$\varkappa$	is the heat conductivity;
β	is the coefficient of volume expansion;
L	is the specific heat of fusion;
r, φ and z	are the cylindrical coordinates;
h	is the thickness of fluid layer;
S	is the surface area;
$g = g\gamma$	is the acceleration of gravity;
mg	is the cylinder weight;
G, Q, M, N, P and R	are the dimensionless parameters of the problem.

Subscripts

Differentiation with respect to time is noted by a dot, and with respect to the coordinate a by a prime.